Numerical local irreducible decomposition

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Abstract. Globally, the solution set of a system of polynomial equations with complex coefficients can be decomposed into irreducible components. Using numerical algebraic geometry, each irreducible component is represented using a witness set thereby yielding a numerical irreducible decomposition of the solution set. Locally, the irreducible decomposition can be refined to produce a local irreducible decomposition. We define local witness sets and describe a numerical algebraic geometric approach for computing a numerical local irreducible decomposition for polynomial systems. Several examples are presented.

Key words and phrases. Numerical algebraic geometry, numerical irreducible decomposition, local irreducible decomposition, numerical local irreducible decomposition

1 Introduction

For a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$, the *algebraic set* defined by f is the set $\mathcal{V}(f) = \left\{x \in \mathbb{C}^N \mid f(x) = 0\right\}$. An algebraic set V is *reducible* if there exist nonempty algebraic sets $V_1, V_2 \subsetneq V$ such that $V = V_1 \cup V_2$ and for $i \neq j$, $V_i \not\subset V_j$. If V is not reducible, it is *irreducible*. For $\mathcal{V}(f)$, there exist irreducible algebraic sets $V_1, \ldots, V_k$, called *irreducible components*, such that $\mathcal{V}(f) = \bigcup_{i=1}^k V_i$ and $V_j \not\subset \bigcup_{i\neq j} V_i$. The irreducible components $V_1, \ldots, V_k$ are said to form the *irreducible decomposition* of $\mathcal{V}(f)$.

A fundamental computation in numerical algebraic geometry is the *numerical irreducible decomposition* (NID), that is, computing a *witness set* for each of the irreducible components; e.g., see [2, Chap. 10]. For an irreducible component $V \subset \mathcal{V}(f) \subset \mathbb{C}^N$ of dimension d and degree r, a witness set for V is the triple $\{f, \mathcal{L}, W\}$ where $\mathcal{L} \subset \mathbb{C}^N$, called a *witness slice*, is a general linear space of codimension d and $W = V \cap \mathcal{L}$, called a *witness point set*, is a set of r points.

One can naturally extend the global notions of reducibility, irreducible components, and irreducible decomposition to the local case (e.g., see [5, Chap. B]). Moreover, one can locally extend to the case that f is holomorphic in an open neighborhood. Our main contribution is to extend the numerical algebraic geometric notions to the local case via *local witness sets* and a *numerical local irreducible decomposition*, defined in Sect. 2, the computation of which is described in Sect. 3, and demonstrated on several examples in Sect. 4 using `Bertini` [1].

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2 Local witness sets

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system, $V_1, \ldots, V_k$ be the irreducible components of $\mathcal{V}(f)$, and $x^* \in \mathcal{V}(f)$. If $x^* \in V_i$, then V_i localized at x^* can be decomposed uniquely, up to reordering, into a finite union of locally irreducible components $T_{i,1}, \ldots, T_{i,m_i}$, e.g., see Theorem 7 of [5, Chap. B]. If $x^* \notin V_i$, then V_i localized at x^* is empty, i.e., $m_i = 0$. Hence, the local irreducible decomposition of $\mathcal{V}(f)$ at x^* is $\bigcup_{i=1}^k \bigcup_{j=1}^{m_i} T_{i,j}$.

Example 1. Consider the irreducible polynomial $f(x) = x_1^2 - x_2^2 + x_2^4$. Hence, for a general $x^* \in \mathcal{V}(f) \subset \mathbb{C}^2$, the irreducible curve $\mathcal{V}(f)$ is locally irreducible at x^*. The origin arises as a self-crossing of the curve $\mathcal{V}(f)$ and hence decomposes into two locally irreducible components at the origin, say

$$T_{1,1}, T_{1,2} = \left\{ \left(x_1, \pm\sqrt{1 - \sqrt{1 - 4x_1^2}}/\sqrt{2} \right) \;\middle|\; x_1 \text{ near } 0 \right\}.$$

As with the global case, where witness sets form the key data structure in formulating a NID, *local witness sets* will be used to formulate a *numerical local irreducible decomposition* (NLID). The two key differences between a witness set and a local witness set, which we formally define below, are:

1. a local witness set is only well-defined on a neighborhood of x^*; and
2. all points in the local witness point set converge to x^* as the witness slice deforms to slice through x^*.

The key to understanding the local structure of an analytic set is the local parameterization theorem (see [5, Chap. C, D, E] and [6]). For a pure d-dimensional reduced analytic set $V \subset \mathbb{C}^N$ containing x^*, the local parameterization theorem implies (among other things) that there is an open ball $\mathcal{U} \subset \mathbb{C}^N$ centered at x^* such that given a general linear projection $\pi : \mathbb{C}^N \to \mathbb{C}^d$ and any open ball $B_\epsilon(\pi(x^*))$ with $\epsilon > 0$ small enough, the map $\pi_{\widehat{V}}$ is a proper branched covering from $\widehat{V} := V \cap \pi^{-1}(B_\epsilon(\pi(x^*))) \cap \mathcal{U}$ onto $B_\epsilon(\pi(x^*))$. Moreover, the sheet number is the multiplicity of the point x^* on V, denoted μ_{x^*}.

Remark 1. Since $\pi_{\widehat{V}}$ is proper, the Remmert proper mapping theorem implies that there is an analytic set $R \subset B_\epsilon(\pi(x^*))$ with $\dim R < d$ such that $\pi_{\widehat{V} \setminus \pi^{-1}(R)}$ is an unbranched μ_{x^*}-sheeted cover from $\widehat{V} \setminus \pi^{-1}(R)$ onto $B_\epsilon(\pi(x^*)) \setminus R$. Hence, if V is locally irreducible at x^*, then $\widehat{V} \setminus \pi^{-1}(R)$ is connected and the monodromy action on any fiber of $\pi_{\widehat{V} \setminus \pi^{-1}(R)}$ is transitive.

The local parameterization theorem is a local version of the Noether Normalization Theorem. For a pure d-dimensional algebraic set $V \subset \mathbb{C}^N$, the Noether Normalization Theorem states that the restriction π_V to V of a general linear projection $\pi : \mathbb{C}^N \to \mathbb{C}^d$ is a proper $\deg V$-to-one map of V onto $\mathbb{C}^d$. Given a general codimension d linear space $\mathcal{L}$ containing x^*, it follows that $\mathcal{L} \cap V$ consists of x^* and $\deg V - \mu_{x^*}$ smooth points. Given a preassigned open set $\mathcal{O}$ around

$\mathcal{L} \cap V$, the intersection of any d codimensional linear space $\mathcal{L}'$ sufficiently near $\mathcal{L}$ will have $\mathcal{L}' \cap V \subset \mathcal{O}$. By choosing $\mathcal{O}$ as the intersection of V with $\deg V - \mu_{x^*} + 1$ disjoint small open balls, we see that the $\mathcal{L}' \cap V$ has precisely μ_{x^*} points near x^*.

Definition 1. *Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a system of functions which are holomorphic in a neighborhood of $x^* \in \mathbb{C}^N$ with $f(x^*) = 0$. Let $V \subset \mathbb{C}^N$ be a locally irreducible component of $\mathcal{V}(f)$ at x^* of dimension d and $\ell_1, \ldots, \ell_d : \mathbb{C}^N \to \mathbb{C}$ be general linear polynomials such that $\ell_i(x^*) = 0$. For $u \in \mathbb{C}^d$, let $\mathcal{L}_u \subset \mathbb{C}^N$ be the linear space defined by $\ell_i(x) = u_i$ for $i = 1, \ldots, d$. A* local witness set *for V is the triple $\{f, \mathcal{L}_{u^*}, W\}$ defined in a neighborhood $U \subset \mathbb{C}^d$ of the origin for general $u^* \in U$ and W is the finite subset of points in $V \cap \mathcal{L}_{u^*}$ which are the start points of the paths defined by $V \cap \mathcal{L}_{u(t)}$ where $u : [0,1] \to U$ is any path with $u(0) = 0$ and $u(1) = u^*$ which converge to x^* as $t \to 0$.*

Remark 2. The choice of points W inside of $V \cap \mathcal{L}_{u^*}$ is well-defined and equal to the multiplicity μ_{x^*} of V at x^*. We call μ_{x^*} the *local degree* of V at x^*.

Remark 3. When V is a curve, the neighborhood U is often referred to as the *endgame operating zone*, e.g., see [2, § 3.3.1]. For all cases, we will call U the *generalized endgame operating zone.*

As Remark 1 suggests, one can perform monodromy loops using local witness sets similarly to classical witness sets. Local witness sets can also be used to sample components and to perform local membership testing.

In particular, a *numerical local irreducible decomposition* consists of a formal union of local witness sets, one for each local irreducible component.

Example 2. Reconsider f from Ex. 1 with $x^* = (0,0)$. For simplicity, we take $\ell_1(x) = x_1$ which then defines the neighborhood $U = \{u \in \mathbb{C} \mid |u| < 1/2\}$. We arbitrarily select $u^* = 1/6$ which implies that

$$\mathcal{V}(f) \cap \mathcal{L}_{u^*} = \left\{ \left(\tfrac{1}{6}, \pm\sqrt{\tfrac{1}{2} - \tfrac{\sqrt{2}}{3}} \right), \left(\tfrac{1}{6}, \pm\sqrt{\tfrac{1}{2} + \tfrac{\sqrt{2}}{3}} \right) \right\}.$$

As u (and hence x_1) deforms to 0, the first two points in $\mathcal{V}(f) \cap \mathcal{L}_{u^*}$ converge to x^* while the last two converge to $(0, \pm 1)$, respectively. For local irreducible components $T_{1,1}$ and $T_{1,2}$ of $\mathcal{V}(f)$ at x^*, local witness sets are

$$\mathcal{W}_1 = \left\{ f, \mathcal{L}_{u^*}, \left\{ \left(\tfrac{1}{6}, \sqrt{\tfrac{1}{2} - \tfrac{\sqrt{2}}{3}} \right) \right\} \right\} \text{ and } \mathcal{W}_2 = \left\{ f, \mathcal{L}_{u^*}, \left\{ \left(\tfrac{1}{6}, -\sqrt{\tfrac{1}{2} - \tfrac{\sqrt{2}}{3}} \right) \right\} \right\},$$

with each $T_{1,i}$ having local degree 1. Since $T_{1,1} \cup T_{1,2}$ form a local irreducible decomposition of $\mathcal{V}(f)$ at x^*, the formal union $\mathcal{W}_1 \cup \mathcal{W}_2$ is a NLID.

3 Computing numerical local irreducible decompositions

When decomposing a pure-dimensional set into its irreducible components, one simplification is to reduce down to the curve case. That is, if $V \subset \mathbb{C}^N$ is pure d-dimensional and $\mathcal{M} \subset \mathbb{C}^N$ is a general linear space of codimension $d-1$, then the irreducible components of V correspond with the irreducible components of $V \cap \mathcal{M}$. Unfortunately, this need not hold for the local case.

Example 3. Consider $V = \mathcal{V}(x_1^2 + x_2^2 + x_3^2) \subset \mathbb{C}^3$ which is irreducible at the origin. For a general complex plane $\mathcal{L} = \mathcal{V}(a_1x_1 + a_2x_2 - x_3)$ through the origin, it is easy to check that $V \cap \mathcal{L}$ consists of two lines through the origin.

The following outlines a procedure for computing a NID that follows from Sect. 2. We assume that we are given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$ and a point $x^* \in \mathcal{V}(f)$. Since we can loop over the irreducible components of $\mathcal{V}(f)$, the key computation is to compute the NLID for an irreducible component $V \subset \mathcal{V}(f)$ given a witness set $\{f, \mathcal{L}, W\}$ for V with $d = \dim V$.

1. Select random linear polynomials $\ell_i : \mathbb{C}^N \to \mathbb{C}$ with $\ell_i(x^*) = 0$.
2. Pick random $u^* \in \mathbb{C}^d$ in the generalized endgame operating zone. Construct the linear spaces $\mathcal{L}_{u^*}$ and $\mathcal{L}_0$ defined by $\ell_i = u_i^*$ and $\ell_i = 0$, respectively. Compute $W' = V \cap \mathcal{L}_{u^*}$ via the homotopy defined by $V \cap (t \cdot \mathcal{L} + (1-t) \cdot \mathcal{L}_{u^*})$.
3. Compute W_{x^*} consisting of points $w \in W'$ such that the path defined by the homotopy $V \cap \mathcal{L}_{t \cdot u^*}$ starting at w at $t = 1$ limit to x^* as $t \to 0$.
4. Use monodromy loops inside the generalized endgame operating zone to compute the local monodromy group which partitions $W_{x^*} = W_1 \sqcup \cdots \sqcup W_s$. The NLID for V at x^* is defined by the formal union $\bigcup_{i=1}^s \{f, \mathcal{L}_{u^*}, W_i\}$.

Remark 4. The key to performing the same computation in the holomorphic case is to compute the finite set W_{x^*} in Item 3. The number of such points in W_{x^*} can be computed via a local multiplicity computation using Macaulay dual spaces [3,9] in certain cases. For example, if $x^* \in \mathbb{C}^N$ and $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$ is a system of holomorphic functions at x^* such that the local dimension of $\mathcal{V}(f)$ at x^* is d, it follows from [4, pg. 158] that the multiplicity of $\{f, \ell_1, \ldots, \ell_d\}$ at x^* is equal to the number of points in W_{x^*}.

4 Examples

4.1 Illustrative example

Consider the irreducible curve $V = \mathcal{V}(x_1^5 + 2x_2^5 - 3x_1x_2(x_1 - x_2)(x_2 - x_1^2)) \subset \mathbb{C}^2$ with Fig. 1(a) plotting the real points of V and $x^* = (0, 0)$. For simplicity, we take $\ell_1(x) = 2x_1 + 3x_2$, $u^* = 1/8$, and $\mathcal{L}_u$ defined by $\ell_1(x) = u$. Hence, $V \cap \mathcal{L}_{u^*}$ consists of five points, with four of the paths defined by the homotopy $V \cap \mathcal{L}_{t \cdot u^*}$ limiting to x^* as $t \to 0$. Therefore, W_{x^*} in Item 3 consists of 4 points.

We now perform monodromy loops which, in the curve case, means looping around 0. We observe that this loop breaks into 3 distinct cycles, two remain on their own branch and two interchange. Therefore, there are 3 local irreducible components as shown in Fig. 1(b), two of local degree 1 and one of local degree 2.

4.2 Local irreducibility and real solutions

If the polynomial system f has real coefficients, the complex conjugate, $\text{conj}(V)$, of an irreducible component $V \subset \mathcal{V}(f)$ is also an irreducible component. If $V \neq \text{conj}(V)$, then then all real points on V must be contained in $V \cap \text{conj}(V)$


(a) (b)

Fig. 1. Plot of (a) the real points of an irreducible quintic curve and (b) the real points near the origin, which locally decomposes into three components.

where $\dim V > \dim(V \cap \text{conj}(V))$. For example, the "home" position of a cubic-center 12-bar mechanism [11], as presented in [10, Fig. 3], can be shown to be rigid, i.e., isolated over the real numbers, by observing that the only two irreducible components containing the "home" position are two sextic curves which are conjugates of each other [7].

The NID is not always sufficient to reveal structure at singularities. Consider the Whitney umbrella $V = \mathcal{V}(x_1^2 - x_2^2 x_3) \subset \mathbb{C}^3$, which is an irreducible surface. For a random point on the "handle," i.e., $x^* = (0, 0, \alpha)$ for random $\alpha \in \mathbb{C}$, the NLID reveals that V at x^* has two local irreducible components, each of local degree 1. At the origin, the NLID reveals that it is irreducible of local degree 2. When $\alpha < 0$, say $x^* = (0, 0, -1)$, global information is not enough to observe that the real local dimension is smaller than the complex local dimension. However, the local viewpoint does indeed reveal that the two local irreducible components are complex conjugates of each other showing a smaller real local dimension.

4.3 Foldable Griffis-Duffy platform

In our last example, we consider the "folded" pose, as shown in [8, Fig. 3], of a foldable Griffis-Duffy platform with the polynomial system available at [1] (see also [2, Chap. 8]). Our local approach verifies that the local irreducible decomposition of the "folded" pose consists of three double lines and a self-crossing of a quartic curve as mentioned in [8,10].

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