Numerical Local Irreducible Decomposition

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Numerical Algebraic Geometry

I would describe Numerical Algebraic Geometry as

the use of numerical tools to study and use zero-sets of polynomials

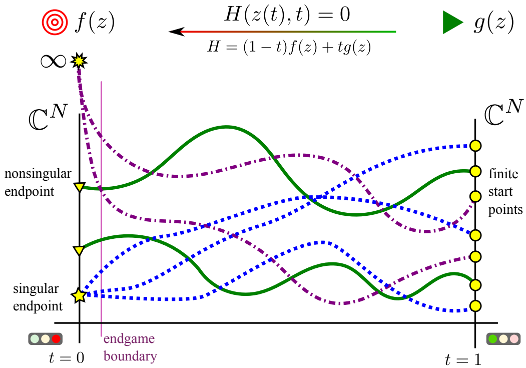
This can involve

- producing the solutions,
- using the solutions for solving a particular math or science problem, and
- the creation of the numerical tools themselves.

Numerical tools can play nicely with many symbolic tools, too!

Homotopy Continuation

The main numerical method used to solve polynomial systems is *homotopy continuation*.



1. Form a homotopy.
2. Track from start time and point to target.

Witness Set

Sometimes polynomial systems don't just have point solutions – a variety can be positive-dimensional.

Numerical Algebraic Geometry describes positive dimensional sets using the *witness set*:

- f – System, possibly randomized or reduced.
- $\mathcal{L}$ – Witness linear slice, dimension-many of them.
- W – Witness points, degree-many of them.

The points are $\cap$ of the linears and system.
The genericity of the slices gives genericity of points. *i.e.* the witness set describes the generic properties of the variety.

Local properties - around generic points

Generic behaviour is generic – random points, and slices in $\mathbb{C}^d$.

If x^* is a random point on $\mathcal{V}(f)$, the variety is locally single-valued over a neighborhood containing it – you can track away from x^*, toward it, and around it, no problem.

$\exists$ a neighborhood of x^* such that slicing nearby yields only one point which limits to x^* as the slice moves to pass through x^*.

The NID already told you everything you needed to know about x^*.

Local properties - around singularities

Singularities are special points. Suppose x^* is a singular point on $\mathcal{V}(f)$. Then some interesting behaviour happens at x^*:

- The usual singular properties:
  - non-invertible Jacobian matrix.
  - tracking becomes difficult nearby.
  - several points come together at x^*.
- And a useful property:
  - possible non-trivial monodromy group generated by walking on a path around x^*.

We'll exploit the monodromy action in the algorithm.

Local Witness set

We introduce a new data type – the local witness set for x^*:

1. f – System, possibly randomized or reduced.
2. $\mathcal{L}_{u^*}$ – Generic slice passing near x^*.
3. W – Witness points, **multiplicity**-many, of them.

- A local witness set is well-defined on a neighborhood of x^*.
- All points in the local witness point set converge to x^* as the witness slice deforms to slice through x^*.
- The number of points in Local Witness Set is the multiplicity of $\mathcal{V}(f)$ at x^*, which we call the *local degree*.

Algorithm - IO

Input:

- $f : \mathbb{C}^N \to \mathbb{C}^n$.
- V, an irreducible component of f.
- x^*, a point on V.

Output:

- The formal union of local witness sets for x^*,

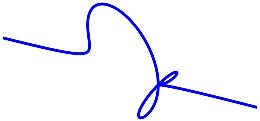
$$\bigcup_{i=1}^{s} \{f, \mathcal{L}_{u^*}, W_i\}$$

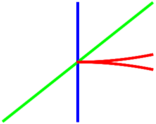
Algorithm - Steps

1. Select random linear polynomials $\ell_i : \mathbb{C}^N \to \mathbb{C}$ with $\ell_i(x^*) = 0$. Pick dimension-many (d) of them.
2. Pick random $u^* \in \mathbb{C}^d$ in the generalized endgame operating zone. Construct the linear spaces $\mathcal{L}_{u^*}$ and $\mathcal{L}_0$ defined by $\ell_i = u_i^*$ and $\ell_i = 0$, respectively. Compute $W' = V \cap \mathcal{L}_{u^*}$ via the homotopy defined by $V \cap (t \cdot \mathcal{L} + (1-t) \cdot \mathcal{L}_{u^*})$.
3. Compute W_{x^*} consisting of points $w \in W'$ such that the path defined by the homotopy $V \cap \mathcal{L}_{t \cdot u^*}$ starting at w at $t = 1$ limit to x^* as $t \to 0$.
4. Use monodromy loops inside the generalized endgame operating zone to compute the local monodromy group which partitions $W_{x^*} = W_1 \sqcup \cdots \sqcup W_s$. The NLID for V at x^* is defined by the formal union $\bigcup_{i=1}^{s} \{f, \mathcal{L}_{u^*}, W_i\}$.

Example – plane curve

$$V = \mathcal{V}(x_1^5 + 2x_2^5 - 3x_1x_2(x_1 - x_2)(x_2 - x_1^2)) \subset \mathbb{C}^2$$





Origin decomposes locally as

- vertical line (blue)
- slanted line (green)
- cusp (red)

Example – Whitney Umbrella

Whitney umbrella

$$V = \mathcal{V}(x_1^2 - x_2^2 x_3) \subset \mathbb{C}^3$$

- For $x^* = (0, 0, \alpha)$, for $\alpha \in \mathbb{C}$, NLID reveals that V at x^* has two local irreducible components, each of local degree 1.
- At the origin, the NLID reveals that it is irreducible of local degree 2.
- When $\alpha < 0$, say $x^* = (0, 0, -1)$, global information is not enough to observe that the real local dimension is smaller than the complex local dimension. However, the local viewpoint does indeed reveal that the two local irreducible components are complex conjugates of each other showing a smaller real local dimension.

Uses

Uses include:

- Local component sampling.
  Generate 1000 random samples near x^*...
- Local membership testing.
  Is $\hat{x}$ *locally a neighbor of* x^**?* ...
- An algorithm for computing tropical curves is almost done, and uses similar method.
- We will also find uses for optimization and other local decompositions.

Thank you for your kind attention

Why can we do this?

The local parameterization theorem, which is a local version of the Noether Normalization Theorem