Advances in Software in Numerical Algebraic Geometry

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Numerical Algebraic Geometry

I would define Numerical Algebraic Geometry as

the use of numerical tools to study and use zero-sets of polynomials

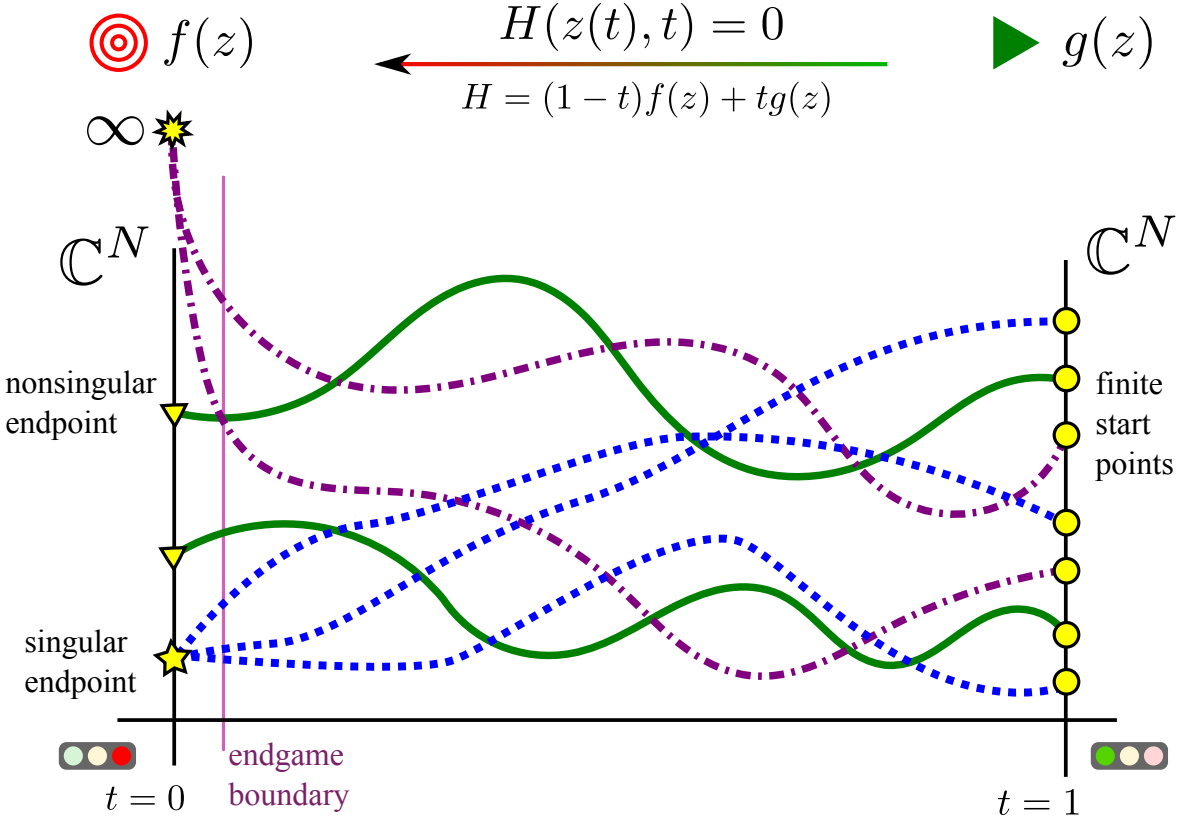
This can involve

- producing the solutions,
- using the solutions for solving a particular math or science problem, and
- the creation of the numerical tools themselves.

Numerical tools can play nicely with many symbolic tools, too.

Homotopy Continuation

The main method used to solve polynomial systems is *homotopy continuation*.



1. Form a homotopy (start system).
2. Track from start to target. Track either from 0 to 1, or the other way around.

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Homotopy Continuators, Solvers

In alphabetic order:

Name	Language	Version	Active
Bertini	C	1.5	☑
Hom4PS-3	C++	3	☑
NAG4M2	Macaulay2	1.7	☑
PHCpack	Ada	2.3.98	☑
Polynomial System Solver	C++	5, preview	☑

All are under active development or maintenance,
and various features.

Bertini

- Authors:
  Dan Bates, Jon Hauenstein, Andrew Sommese, Charles Wampler.
- Features:
  Adaptive multiple precision, MPI parallelism, m-homogeneous start system, user-defined homotopies, Numerical Irreducible Decomposition, sampling of components, deflation of singular components, endgame methods for singular solutions.
- Interfaces:
  Macaulay2, Singular, Matlab. Mathematica incoming.
- Under active re-development into C++, with bindings and support for Python scripts.
- Date of original release: 2005
- License: Almost Open. Bertini2 will be GPL V3.
- Homepage: `bertini.nd.edu`

Hom4PS-3

- Authors:
  Tianran Chen, T.Y. Li, Tsung-Lin Lee.
- Features:
  Automatic multiple precision, MPI parallelism, GPU acceleration, Total degree homotopy, Cheater's homotopy, Polyhedral homotopy, Complex Newton's homotopy, singular endgame.
- News from Tianran about upcoming Hom4PS-4:
  "GPU computation is playing a major role."
  "Tropical geometry has inspired new techniques for mixed volume computation."
- Date of Hom4PS-2 release: 2008. Version 3 out now, version 4 under development.
- License: Proprietary
- Homepage: `http://www.hom4ps3.org/`

NAG4M2

- Authors:
  Anton Leykin, Robert Krone.
- Features:
  Tracking of homotopies, start system formation, NID, sampling, membership testing, Newton-Raphson method, certified tracking, Scheme analysis, .
- Interfaces:
  PHCpack, Bertini.
- Date of original release: Journal-published in 2011.
- License: GPL V2 or V3
- Source homepage: `http://www.math.uiuc.edu/Macaulay2/`
  `http://people.math.gatech.edu/~aleykin3/NAG4M2/`

PHCpack

- Authors:
  Jan Verschelde.
- Features:
  Many-core GPU acceleration, scaling of systems, double double and quad double arithmetic, arbitrary precision arithmetic, linear start system, Polyhedral homotopy.
- Interfaces:
  Maple, Matlab, C, Python, Sage, Macaulay2
- Date of original release: 1995
- License: GPL V3
- Source homepage: `https://github.com/janverschelde/PHCpack`

PSS5

- Authors:
  Gregorio Malajovich.
- Features:
  Mixed volume computations, sparse condition numbers, condition-metric homotopy, tropical curves.
- Date of preview release: May 2015
- License: GPL V3
- Source homepage: `http://sourceforge.net/projects/pss5/`

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Certification

Pure symbolic methods have the trust of mathematicians and scientists alike, but can be prohibitively expensive in terms of computational cost. Hence, the role of numerical methods.

What can be done about the trust issue?

Certification is a process by which a numerical result can be proven to be correct.

- *a priori* – the hardest kind. If the computation terminates, it will be correct.
- *a posteriori* – obtain a heuristic result, and prove it is correct.

Cadenza

- Paper: Hauenstein, Haywood, Liddell. *An a posteriori certification algorithm for Newton homotopies.*
- Software: Cadenza : `https://www3.nd.edu/~aliddel1/research/cadenza/`
- A Newton homotopy is a homotopy of the form

$$H(x,t) = f(x) + tv$$

  where $v \in \mathbb{C}^n$, and t runs from 1 to 0.
- Run the Newton homotopy, store all x and t values, and *then* certify that the path was / not continuous.
- To certify step was on continuous path, show that each pair of points on the path satisfies two conditions. Use criteria from Smale's α theory.

Certified Newton homotopies

Hauenstein, Liddell. *Certified predictor-corrector tracking for Newton homotopies.* 2015

- *A priori* certified Newton homotopy tracking.
- Bounds the number of steps needed over the track.
- One constant predictor step, followed by multiple corrector steps.
- Euler step, followed by corrector step.
- Narrows the gap between pure heuristics and *a priori* certification.
- Needs to be fully implemented in a deployed software package.

Real solution sets

Brake, Hauenstein, Liddell. *Numerically certifying the completeness of real solution sets to polynomial systems.* Under preparation.

Q: How can you know whether you have found all the real solutions?
A: Use Sums of Squares (SOS) to prove that a set of interpolants is the real radical ideal.

Come see the talk by Alan Liddell later in this minisymposium –
Thursday at 11:30.

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Current Trends

- GPU acceleration. Using 'massively parallel' architectures.
- Better and faster start system creation. Continuing to reduce runtime of solves.
- Program interfaces. Using one program from another.
- Newton homotopies. Using a time-space de-coupled system to generate solutions quickly, even for non-polynomial systems.

GPU

Some recent papers and advances on the GPU front:

- Verschelde. *Accelerating Polynomial Homotopy Continuation on a Graphics Processing Unit with Double Double and Quad Double Arithmetic.* 2015
- Verschelde. *GPU acceleration of Newton's method for large systems of polynomial equations in double double and quad double arithmetic.* 2014
- Verschelde, Yu. *Tracking Many Solution Paths of a Polynomial Homotopy on a Graphics Processing Unit.* 2015
- Chen. Hom4PS-4 will include GPU processing.

Start systems, root counts, and tracking

Some recent advances in start systems:

- Chen, Li, Lee. *Mixed volume computation in parallel.* 2014

Root counting:

- Malajovich. *Computing mixed volume and all mixed cells in quermassintegral time.* 2014
- Malajovich. *On the expected number of zeros of nonlinear equations.* 2013
- Emiris, Vidunas. *Root counts of semi-mixed systems, and an application to counting nash equilibria.* 2014
- Bates, Hauenstein, Niemerg, Sottile. *Software for the Gale Transform of fewnomial systems and a Descartes rule for fewnomials.* Submitted 2015

Tracking

- Muñoz, Hernandez-Martinez, Vásquez-Leal. *Spherical Continuation Algorithm with Spheres of Variable Radius to Trace Homotopy Curves.* 2015

Interfaces

Some recent advances in program interfaces:

- Verschelde. *Modernizing PHCpack through phcpy.* 2013
  PHCpack $\Longleftrightarrow$ Python.
- Gross, Petrovic, Verschelde. *Interfacing with PHCpack.* 2013
  PHCpack $\Longleftrightarrow$ Macaulay2.
- Bates, Newell, Niemerg. *BertiniLab: A MATLAB interface for solving systems of polynomial equations.* 2015
  Bertini $\Longleftrightarrow$ Matlab.
- Bates, Gross, Leykin, Rodriguez. *Bertini for Macaulay2.* 2013
  Bertini $\Longleftrightarrow$ Macaulay2

Newton homotopy

Some recent advances in Newton homotopies:

- Mehta, Chen, Hauenstein, Wales. *Newton homotopies for sampling stationary points of potential energy landscapes*. 2014
- Mehta, Chen, Morgan, Wales. *Exploring the potential energy landscape of the Thomson problem via Newton homotopies*. 2015
- Hauenstein, Liddell. *Certified predictor-corrector tracking for Newton homotopies*. 2014
- Hauenstein, Haywood, Liddell. *An a posteriori certification algorithm for Newton homotopies*. 2014

Fixed-point homotopies:

- Chen, Mehta. *An index-resolved fixed-point homotopy and potential energy landscapes*. 2015

A fun application:

- Guo, Xia, Du, Ji, Han. *Research of critical ambient temperature of cylindrical fireworks and crackers*. 2014

Thank you for your kind attention